Field-Particle Dynamics in Spacetime Geometries

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Abstract

With the aid of a Fermi-Walker chart associated with an orthonormal frame attached to a time-like curve in spacetime, a discussion is given of relativistic balance laws that may be used to construct models of massive particles with spin, electric charge and a magnetic moment, interacting with background electromagnetic fields and gravitation described by non-Riemannian geometries. A natural generalisation to relativistic Cosserat media is immediate.

1 Introduction

The language of relativistic fields on a spacetime offers, probably, the most succinct and complete description of the basic laws of physics. From these laws we expect to accommodate the enormous amount of data derived from experiment, predict new phenomena and modify or reject laws in accordance with observation. Of necessity this process invariably involves some level of approximation. As experimental technology improves it becomes possible to refine such approximations and thereby seek more detailed experimental verification of theoretical predictions.

A field description of relativistic gravitation was formulated by Einstein almost a century ago. Since then it has proved one of the most successful descriptions of large scale gravitational phenomena in all of physics and has profound implications for the notions of space and time on all scales. Nevertheless this theory of gravitation cannot be complete and it poses many challenges in both classical and quantum domains. Modern experimental tools of increasing sophistication are now being brought to bear on some of these problems with the search for classical gravitational waves, the breakdown of the "equivalence principle" and the influence of gravitation on the "collapse of the wave-function". Along with these technological advances

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new theoretical avenues continue to be explored in which all the basic interactions between matter are subsumed into a coherent framework.

Viable approximation schemes that link the results of experimental probes and theoretical descriptions of the basic interactions will remain as bridges between these continuing developments. Since the early days of relativity it was recognised that the derivation of the motion of extended "matter" in gravitational fields was a non-trivial problem and great effort was expended in deriving effective approximations. The work of Einstein, Infeld, Hoffman, Mathisson, Papapetrou, Pryce , Tulczyjew and others [1, 2, 3, 4, 5] culminated in a series of papers by Dixon, [6, 7, 9, 10, 11, 12], Ehlers and Rudolph [14] and Kunzle [8], where it was demonstrated that a viable scheme could be established, given suitable subsidiary conditions, in which an exact field description could be replaced by an approximation procedure in which extended matter was regarded at each order of approximation by a finite set of (non-unique) matter and electromagnetic multipoles. Such a scheme offers in principle a viable approach to the motion of extended matter and deserves to be taken seriously in the context of modern experiments that purport to analyse the fundamentals of the gravitational interaction with spinning matter. However theories of gravitation with spinning matter sit somewhat un-naturally in the pseudo-Riemannian (i.e. torsion free) environment in which the whole Dixon framework was formulated. Furthermore it is not necessary to have "spinning sources" to accommodate gravitational fields with torsion. Indeed a Lorentz gauge-covariant formulation of the Brans-Dicke theory naturally yields a metric-compatible connection on the bundle of linear frames over spacetime with torsion determined by the gradient of the Brans-Dicke scalar field [15]. Since many modern variants of this theory (and various "string theories") abound with scalar gravitational fields one should seriously consider the possibility that gravitation may have a torsional component that is absent (by hypothesis) in the pseudo-Riemannian framework given by Einstein.

The multipole analysis formulated by Dixon relies heavily on the use of bi-tensors and tensor densities and the pseudo-Riemannian structure of spacetime. To extend this methodology to general spacetime structures a new approach is adopted here based on the use of differential forms, structure equations, a (generalised) Fermi chart associated with a timelike worldline and the transport of a Fermi-Walker frame along spacelike autoparallels of a metric-compatible connection with torsion. This approach may be contrasted with the use of adapted coordinates employed by Tulczyjew and Tulczyjew [13] in a torsion-free context. Although they employ a Fermi chart in discussing the equations of motion of an electrically neutral particle, their methods, detailed calculations and conclusions differ significantly from the results to be discussed below even in the case considered, where the torsion is zero. In particular their resulting equations admit unnatural solutions in Minkowski spacetime as will be discussed in section VIII.

The approach to be considered here considers the modelling of relativistic massive particle interactions with background electromagnetic and gravitational fields with possible torsion. As stressed above it is assumed that the *underlying* laws describing all phenomena can be formulated in terms of tensor and spinor *fields* over a manifold. Depending on the physical scales involved such fields are in general subject to either the laws of classical or quantum field theory. In this paper emphasis is on classical fields. In particular it is assumed that classical gravitation can be described in terms of a geometry on a spacetime and the electromagnetic field is a closed 2-form on this manifold. Like Newtonian gravitation, electromagnetism is distinguished from other gauge fields by its long range character. It can also be related to the curvature of a connection on a principle $U(1)$ bundle over spacetime. The connection on the bundle of linear frames over spacetime will be that induced from the bundle or orthonormal frames. Thus the local gauge group for gravitation may be regarded as the covering of the Lorentz group $SO(1,3)$. Shorter range 2-form (internal symmetry non-Abelian Yang-Mills) fields, regarded as curvatures associated with connections on other principle group bundles are also responsible for the basic interactions in Nature. Matter fields (regarded as sections of associated bundles) carry representations of these Yang-Mills groups and their gauge covariant interactions with gauge fields are responsible for the "charges" of massive field quanta. All fields influence the geometry of spacetime and in turn the gravitational field influences the behavior of all other fields. The field laws for the coupled system of gauge fields and matter may be compactly derived as differential equations that render extremal the integral of some (action) 4-form on spacetime. This integral is constructed to be invariant under the action of all local gauge groups. The resulting classical variational field equations are thereby covariant under changes of section induced by associated local changes of frame (gauge). In this article the Yang-Mills internal symmetry group made explicit will be restricted to electromagnetism. It is the gauge invariance of some action integral that is of paramount importance in the developments below.

2 Balance Laws

Classical *point particles* are an abstraction. In general they offer a useful approximation to the behavior of localised matter and form the basis of rigid body dynamics in Newtonian physics following the laws of Newtonian particle dynamics. The behavior of Newtonian deformable continua require new laws that exploit the high degree of Killing symmetry in spacetime devoid of gravitation where matter, (observed from weakly accelerating reference frames) has a small speed relative to that of light. The *balance laws* of energy, linear and angular momentum must be separately postulated in order

to provide a framework for the description of non-relativistic deformable continua. However the evolution of such continua can only be determined from such balance laws once a choice of constitutive relations (Cauchy stress tensor) consistent with the laws of thermodynamics has been made. Such relations serve to render the balance laws predictive and are either phenomenological in nature or arise as coarse-grained approximations applied to a statistical or quantum analysis that recognises the atomistic nature of matter. Newtonian balance laws refer to arbitrary distributions of matter in space. Point particle and rigid-body dynamics arise as zeroth and first order multipole approximations [47]. There are other types of approximation that are useful when the continuum is slender in either one or two dimensions relative to the third (i.e. rods and shells). Furthermore the internal state of such structures can be expressed as degrees of freedom associated with a field of frames in the medium (Cosserat models). In all such cases a successful phenomenological description is sought in terms of a small number of physical parameters (mass, moment of inertia tensor, elastic moduli, pressure, density, thermal and viscosity coefficients...) that is amenable to experimental verification. For extended structures, such as fluids or localised structures such as strings or rods it is the relatively weak interaction of the continuum with its environment that renders such a description effective and enables one to distinguish externally prescribed forces and constraints from the responses that they produce.

An analogous strategy is sought to describe the *relativistic* motion of localised matter from an underlying field theory involving gravitation and electromagnetism. Since the matter fields may carry representations of the Lorentz group (or its covering) as well as $U(1)$ one expects to accommodate electrically charged particles with some kind of spin. The basic approximation considered here treats the gauge fields and gravitation as a background that is prescribed independently of the matter fields. These on the other hand are required to satisfy gauge-covariant field equations in the presence of the prescribed gauge fields and gravitation. The metric of the background spacetime will, however, possess no particular isometries to provide the analogues of the Newtonian balance laws. Nevertheless the principle of local gauge covariance for the interactions between fields provides powerful dynamical constraints on the field variables that arise in any variational description. This observation is well known and its implications can be found in many parts of the literature. Such constraints will be explored below in the context of adapted orthonormal frame fields and associated coordinates based on a Fermi chart associated with a time-like curve in spacetime.

The equations of motion proposed by Dixon offer a consistent dynamical scheme [14] for the classical behavior of ““spinning matter” [16, 17] in pseudo-Riemannian spacetimes. In the lowest pole-dipole approximation they can be solved analytically for electrically neutral particles in gravitational fields with high symmetry [18] but in general recourse to numerical

methods is required [19], [20, 21]. We shall recover these equations below in spaces with zero torsion albeit without recourse to bi-tensors and tensor densities. Furthermore we shall show that in background spacetimes in which gravitation is described by geometries having connections with torsion, electrically charged spinning particles deviate from such histories due to prescribed interactions between the background fields and the spin and charge of the particle.

We argue that in more general spacetimes even electrically neutral spinless particles may have histories that follow autoparallels of connections with torsion instead of geodesics of the spacetime metric.

3 Fermi Coordinates

Let M be a smooth manifold with a metric tensor g and a metric compatible connection ∇. Denote the tangent space to M at point p by T_pM. For any p and any vector v in T_pM let $\gamma_v : \lambda \mapsto \gamma_v(\lambda) \in M$ be the unique autoparallel ($\nabla_{\gamma_v{}'}\gamma_v{}' = 0$) of ∇ with

$$\gamma_v(0) = p \tag{1}$$

$$\gamma_v'(0) = v. \tag{2}$$

Provided $\gamma_v(1)$ exists denote it by $\exp_p(v)$ i.e.

$$\exp_p(v) \equiv \gamma_v(1) \in M. \tag{3}$$

The exponential map $\exp_p : T_pM \mapsto M$ is therefore defined in a neighbourhood of $0 \in T_pM$ and is in fact a diffeomorphism [22, 23] .

Suppose M is a four dimensional spacetime and g has Lorentzian signature $(-,+,+,+)$. The geometry of M is described by g and ∇ which is metric-compatible but not assumed to be torsion free. Let $\sigma : t \in \mathbf{R}^+ \mapsto M$ be a time-like future pointing affinely parametrised curve ($g(\dot\sigma,\dot\sigma) = -1$) and $\hat{\mathcal{F}} \equiv \{\hat X_0, \hat X_1, \hat X_2, \hat X_3\}$ a g-orthonormal frame on σ with $\dot\sigma(t) = \hat X_0|_{\sigma(t)}$. Then a Fermi chart Ψ with Fermi coordinates (t, x^1, x^2, x^3) relative to the curve σ, frame $\hat{\mathcal{F}}$ and arbitrary point $\sigma(t_0)$ on the image of σ is defined by:

$$\Psi^0(\exp_{\sigma(t)} v) = t - t_0 \tag{4}$$

$$\Psi^k(\exp_{\sigma(t)} v) = x^k \tag{5}$$

where $v = \sum_{j=1}^3 x^j \hat X_j \in T_{\sigma(t)}M$. In the following repeated Latin indices will be summed from 1 to 3 while repeated Greek indices will be summed

from 0 to 3. The chart is defined on any open neighbourhood $\mathcal{U}$ of σ containing points that can be joined to σ by an affinely parameterised unique autoparallel meeting σ orthogonally. Such a definition gives rise to a class of Fermi charts for a given curve σ with frames related to each other by local $SO(3)$ transformations and translations of origin $\sigma(t_0)$ along σ [24]. In any particular chart with coordinates $\{t, x^k\}$ one has the local vector fields $\{\partial_t, \partial_k \equiv \frac{\partial}{\partial x^k}\}$ on $\mathcal{U}$ which by construction are g-orthonormal on σ. Furthermore, by definition, the space-like autoparallels γ_{∂_k} that leave σ at $\{t, 0, 0, 0\}$ can be affinely parameterised in this chart by the equations

$$t - t_0 = \tau \tag{6}$$

$$x^k = p^k \lambda \tag{7}$$

where $(p^1)^2 + (p^2)^2 + (p^3)^3 = 1$. Henceforth the origin is fixed with $t_0 = 0$. The parameters p^k are the direction cosines of the tangent to such an autoparallel, with respect to $\hat{\mathcal{F}}$ at $\{t, 0, 0, 0\}$. It will prove useful below to introduce Fermi-normal hyper-cylindrical polar coordinates $\{\tau, \lambda, p^k\}$ where $(p^k\, p^k) = 1$ on part of $\mathcal{U}$. These are naturally related to the coordinates $\{t, x^k\}$ by the transformation $t = \tau$ and $x^k = p^k \lambda$. Thus any function f of t, x^1, x^2, x^3 can be regarded as a function F of $t, \lambda, p^1, p^2, p^3$ on a suitable domain

$$f(t, x^1, x^2, x^3) = f(t, \lambda p^1, \lambda p^2, \lambda p^3) = F(\tau, \lambda, p^1, p^2, p^3). \tag{8}$$

Using the inverse relations

$$\lambda^2(x^1, x^2, x^3) = x^j x^j \tag{9}$$

$$p^k(x^1, x^2, x^3) = \frac{x^k}{\lambda(x^1, x^2, x^3)} \tag{10}$$

where $\lambda \neq 0$ one has the directional derivative

$$\partial_\lambda = p^k \partial_k \tag{11}$$

and the relations $\partial_\lambda p^k = 0$ and $\frac{\partial p^k}{\partial p^j} = -\frac{p^j}{p^k}$ for $j \neq k$. If F is analytic in λ:

$$F(\tau, \lambda, p^1, p^2, p^3) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \, (\partial^k{}_\lambda F)(\tau, 0, p^1, p^2, p^3) \tag{12}$$

$$= \sum_{k=0}^{\infty} \sum_{i_1, i_2 \ldots i_k = 1}^{3} \frac{\lambda^k}{k!} \, p^{i_1} \ldots p^{i_k} \, (\partial_{i_1} \ldots \partial_{i_k} f)(t, 0, 0, 0) \tag{13}$$

Such expansions will be used to represent the structure 0-forms and 1-forms to be introduced below and partial derivatives of F with respect to λ will be denoted F'. To eliminate excessive notation, expressions evaluated on the curve σ will be understood to be represented in the coordinate system $\{t, x^k\}$ and the abbreviation $\hat{f}$ often used for f restricted to $\sigma(t)$.

4 Ortho-normal frames on $\mathcal{U}$

The coordinate frame $\mathcal{F} \equiv \{\partial_t, \partial_k\}$ on $\mathcal{U}$ is only orthonormal on σ where it coincides with $\hat{\mathcal{F}}$. However one can displace the frame $\hat{\mathcal{F}}$ by parallel transport along the space-like autoparallel curves that emanate from σ using the connection ∇. Since this connection is assumed metric-compatible this process will give rise to a field of orthonormal frames $\mathcal{O}$ on $\mathcal{U}$ [25]. The next task is to calculate the orthonormal coframe field $\{e^\alpha\}$ dual to $\mathcal{O}$ in terms of the coframe $\{d\,t, d\,x^k\}$ dual to $\mathcal{F}$. If one assumes that the curvature and torsion are analytic in the vicinity $\mathcal{U}$ of σ this can be done by recursively differentiating the structure equations with respect to the radial variable λ along a fixed spacelike autoparallel $\gamma_{\partial_\lambda}$ on $\mathcal{U}$.

At each step of the process one evaluates the (derivatives of) structure forms at $\lambda = 0$ thereby building up a Taylor series in λ for each. To initialise the process one must fix the zeroth order terms in the expansion. The coframe field $\{e^\alpha\}$ along $\gamma_{\partial_\lambda}$ is defined to satisfy

$$\nabla_{\gamma'_{\partial_\lambda}} e^\alpha = 0 \tag{14}$$

i.e

$$i_{\partial_\lambda} \omega^\alpha{}_\beta = 0 \tag{15}$$

in terms of the interior derivative of the connection 1-forms of ∇ in the coframe $\mathcal{O}$. Since all points of $\mathcal{U}$ can be connected by such autoparallels we conclude that $\omega^\alpha{}_\beta$ must be independent of $d\,\lambda$ in the frame $\mathcal{O}$ and in particular $\hat{\omega}^\alpha{}_\beta(\hat{\partial}_j) = 0$ on σ. Since $\hat{e}^i$ is dual to $\hat{X}_j = \hat{\partial}_j$ then $\hat{e}^i = \hat{dx^i}$. But $d\,x^i = p^i\,d\,\lambda + \lambda\,d\,p^i$ so e^i pulls back to $p^i\,d\lambda$ at $\lambda = 0$. Also since $\dot{\sigma} = \frac{\hat{\partial}}{\partial t}$ then $\hat{e}^0 = \hat{d}t$. In general

$$e^\alpha = e^\alpha(\partial_t)\,d\,t + e^\alpha(\partial_\lambda)\,d\,\lambda + e^\alpha(\frac{\partial}{\partial p^k})\,d\,p^k. \tag{16}$$

But along $\gamma_{\partial_\lambda}$, $\partial_\lambda(e^\alpha(\partial_\lambda)) = \nabla_{\partial_\lambda}(e^\alpha(\partial_\lambda)) = 0$, since e^α is parallel and $\gamma_{\partial_\lambda}$ is an autoparallel. Again, we may conclude that on $\mathcal{U}$ the structure function $e^\alpha(\partial_\lambda)$ is independent of the coordinate λ. It follows that $e^i(\partial_\lambda) = e^i(\partial_\lambda)|_\sigma = p^i$ and $e^0(\partial_\lambda) = e^0(\partial_\lambda)|_\sigma = \hat{d}t(\partial_\lambda)|_\sigma = 0$. Thus one may write:

$$e^\beta = \mathcal{F}^\beta\;d\,t + \Pi^\beta\,d\,\lambda + \mathcal{E}^\beta \tag{17}$$

where $\mathcal{F}^{\beta}$ are 0 forms dependent on t, λ, p^k and

$$\Pi^0 = 0,\ \Pi^k = p^k \tag{18}$$

$$\mathcal{E}^{\alpha} = \delta^{\alpha}_{0}\,\mathcal{Y} + \delta^{\alpha}_{k}\,\mathcal{Q}^k \tag{19}$$

with 1-forms $\mathcal{Y}$, $\mathcal{Q}^k$ independent of $d\,t$, $d\,\lambda$ but components dependent on t, λ, p^k. Thus:

$$e^0 = \mathcal{F}^0\ d\,t + \mathcal{Y}_j\,d\,p^j \tag{20}$$

$$e^k = \mathcal{F}^k\ d\,t + p^k\,d\,\lambda + \mathcal{Q}^k{}_j\,d\,p^j. \tag{21}$$

where

$$\hat{\mathcal{F}}^0 = 1 \tag{22}$$

$$\hat{\mathcal{Y}} = \hat{\mathcal{Q}}^k = \hat{\mathcal{F}}^k = 0. \tag{23}$$

Similarly, since $\omega^{\alpha}{}_{\beta}(\partial_{\lambda}) = 0$ on $\mathcal{U}$, $\omega^{\hat{\alpha}}{}_{\beta}(\hat{\partial}_j) = 0$. Using $\frac{\partial p^j}{\partial\lambda} = 0$ and

$$\frac{\partial}{\partial p^j} = \lambda\frac{\partial}{\partial x^j} - \lambda p^j \sum_{k\neq j}\frac{1}{p^k}\frac{\partial}{\partial x^k} \tag{24}$$

one may write

$$\omega^{\alpha}{}_{\beta} = \mathcal{A}^{\alpha}{}_{\beta}\ d\,t + \mathcal{C}^{\alpha}{}_{\beta} \tag{25}$$

where $\mathcal{A}^{\alpha}{}_{\beta}$ are 0 forms dependent on t, λ, p^k and $\mathcal{C}^{\alpha}{}_{\beta}$ are 1-forms independent of $d\,t$, $d\,\lambda$ but with components dependent on t, λ, p^k.

Furthermore

$$\omega^{\alpha}{}_{\beta}(\partial_t)|_{\sigma(t)} = \hat{\mathcal{A}}^{\alpha}{}_{\beta} \tag{26}$$

$$\omega^{\alpha}{}_{\beta}(\partial_{p^j})|_{\sigma(t)}\ d\,p^j = \hat{\mathcal{C}}^{\alpha}{}_{\beta} = \omega^{\alpha}{}_{\beta}(\partial_{x^j})\ d\,p^j\,|_{\sigma(t)} = 0. \tag{27}$$

The structure functions $\hat{\mathcal{A}}^{\alpha}{}_{\beta}$ can be determined in terms of properties of the frame $\hat{\mathcal{F}}$ on σ. Suppose the elements of $\hat{\mathcal{F}}$ satisfy:

$$F_{\dot{\sigma}}\hat{X}_{\alpha} = \Omega_{\alpha}{}^{\beta}(\tau)\hat{X}_{\beta} \tag{28}$$

where $\Omega_{\alpha\beta} = -\Omega_{\beta\alpha}$, $\Omega_{0j} = 0$ and $\Omega_{ij} = \epsilon_{ij}{}^{k}\,\Omega_k(\tau)$ in terms of the alternating symbol ϵ^{ijk}, indices on Ω and ϵ being raised or lowered with the Lorentzian metric components $\eta = diag(-1, 1, 1, 1)$. The (generalised) Fermi-Walker derivative [26] above is defined by:

$$F_{\dot\sigma}\hat X_\alpha = \nabla_{\dot\sigma}\hat X_\alpha + g(\dot\sigma, \hat X_\alpha)\mathcal{A}_\sigma - g(\mathcal{A}_a, \hat X_\alpha)\dot\sigma \tag{29}$$

$$= \hat\omega^{\mu}{}_{\alpha}(\dot\sigma)\hat X_\mu + g(\dot\sigma, \hat X_\alpha)\mathcal{A}_\sigma - g(\mathcal{A}_\sigma, \hat X_\alpha)\dot\sigma \tag{30}$$

where $\mathcal{A}_\sigma \equiv \nabla_{\dot\sigma}\dot\sigma$. Such a transport law maintains $\hat X_0$ tangent to σ and $\hat{\mathcal{F}}$ g-orthonormal even though σ need not be an autoparallel of ∇. So given $\dot\sigma = \hat X_0 \equiv \hat\partial_t$ with $\hat e^{\nu}(\hat X_\mu) = \delta^{\nu}_{\mu}$ application of $\hat e^{\nu}$ to (28) yields:

$$\hat\omega^{\nu}{}_{\alpha}(\hat\partial_t) = \Omega_{\alpha}{}^{\nu} - \eta_{0\alpha}\,\hat e^{\nu}(\mathcal{A}_\sigma) + g(\mathcal{A}_\sigma, \hat X_\alpha)\,\delta^{\nu}_{0} \tag{31}$$

or

$$\hat\omega^{0}{}_{i}(\hat\partial_t) = g(\mathcal{A}_\sigma, \hat X_i) \tag{32}$$

$$\hat\omega^{i}{}_{j}(\hat\partial_t) = \Omega_i{}^{j}(\tau) = \Omega_{ij}(\tau). \tag{33}$$

Thus

$$\hat A^{\alpha}{}_{\beta} = \hat\omega^{\alpha}{}_{\beta}(\hat\partial_t) \equiv \Lambda^{\alpha}{}_{\beta} \tag{34}$$

say, is fixed in terms of the orthornormal components of the *acceleration* $\mathcal{A}_\sigma$ of σ and the *instantaneous angular velocity* Ω_k of the frame $\hat{\mathcal{F}}$ along σ. Since $\dot\sigma$ is normalised and time-like the acceleration is orthogonal to $\dot\sigma$. It is convenient to denote the acceleration by $\mathbf{A}$ and write

$$\mathbf{A}_i = g(\mathcal{A}_\sigma, \hat X_i). \tag{35}$$

If one chooses $\Omega_k = 0$ then $\hat{\mathcal{F}}$ is said to be a non-rotating frame along σ. Thus with $\hat{\mathcal{C}}^{\alpha}{}_{\beta} = 0$ and $\hat{\mathcal{A}}^{\alpha}{}_{\beta} = \Lambda^{\alpha}{}_{\beta}$ the equations (20), (21), (141) subject to (27), (34), (23), (22) can be substituted into the structure equations [27, 22, 23] that define the torsion and curvature of ∇:

$$d\,e^{\alpha} = -\omega^{\alpha}{}_{\beta}\wedge e^{\beta} + T^{\alpha} \tag{36}$$

$$d\,\omega^{\alpha}{}_{\beta} = -\omega^{\alpha}{}_{\gamma}\wedge\omega^{\gamma}{}_{\beta} + R^{\alpha}{}_{\beta} \tag{37}$$

where $T^\alpha = \frac{1}{2}\, T^\alpha{}_{\mu\nu}\, e^\mu \wedge e^\nu$ and $R^\alpha{}_\beta = \frac{1}{2}\, R^\alpha{}_{\beta\mu\nu}\, e^\mu \wedge e^\nu$ are the torsion and curvature 2-forms in the frame $\mathcal{O}$. Equations (36), (37) must be satisfied for all λ along auto-parallels that leave $\sigma(t)$ in any direction $\{p^j\}$ so by equating forms containing $d\,\lambda \wedge d\,t$ and $d\,\lambda \wedge d\,p^k$ on each side one can find differential equations for the structure forms $\mathcal{F}^\alpha$, $\mathcal{Y}$, $\mathcal{Q}^i$, $\mathcal{A}^\alpha{}_\beta$, $\mathcal{C}^\alpha{}_\beta$:

$$\mathcal{F}^{\alpha\,\prime} = \mathcal{A}^\alpha{}_k\, p^k + T^\alpha{}_{k\mu}\, \mathcal{F}^\mu\, p^k \tag{38}$$

$$\mathcal{Y}' = \mathcal{C}^0{}_k\, p^k + T^0{}_{k0}\, p^k\, \mathcal{Y} + T^0{}_{kn}\, p^k\, \mathcal{Q}^n \tag{39}$$

$$\mathcal{Q}^{i\,\prime} = d\,p^i + \mathcal{C}^i{}_k\, p^k + T^i{}_{k0}\, p^k\, \mathcal{Y} + T^i{}_{kn}\, p^k\, \mathcal{Q}^n \tag{40}$$

$$\mathcal{A}^\alpha{}_\beta{}' = R^\alpha{}_{\beta k\mu}\, p^k\, \mathcal{F}^\mu \tag{41}$$

$$\mathcal{C}^\alpha{}_\beta{}' = R^\alpha{}_{\beta k0}\, p^k\, \mathcal{Y} + R^\alpha{}_{\beta kn}\, p^k\, \mathcal{Q}^n\,. \tag{42}$$

These have unique solutions satisfying the conditions:

$$\hat{\mathcal{F}}^0 = 1 \tag{43}$$

$$\hat{\mathcal{Y}} = 0,\;\; \hat{\mathcal{Q}}^k = 0,\;\; \hat{\mathcal{F}}^k = 0,\;\; \hat{\mathcal{C}}^\alpha{}_\beta = 0 \tag{44}$$

$$\hat{\mathcal{A}}^\alpha{}_\beta = \Lambda^\alpha{}_\beta\,. \tag{45}$$

By successively differentiating the differential equations above with respect to λ and evaluating the results at $\lambda = 0$ one may construct the Taylor series representation of each analytic structure form about $\lambda = 0$ in terms of (derivatives of) the components of the curvature and torsion tensor and the transport functions $\Lambda^\alpha{}_\beta$ on σ. The first few iterations of this process can be readily established. To first order in λ one finds:

$$\hat{\mathcal{F}}^{\alpha\,\prime} = p^k\,(\Lambda^\alpha{}_k + \hat{T}^\alpha{}_{k0}) \tag{46}$$

$$\hat{\mathcal{Y}}' = 0 \tag{47}$$

$$\hat{\mathcal{Q}}^{i\,\prime} = \,d\,p^i \tag{48}$$

$$\hat{\mathcal{A}}^{\alpha}{}_{\beta}{}' = \hat{R}^{\alpha}{}_{\beta k0}\; p^k \tag{49}$$

$$\hat{\mathcal{C}}^{\alpha}{}_{\beta}{}' = 0 \tag{50}$$

Radial derivatives of the curvature and torsion start to appear at the next order:

$$\hat{\mathcal{F}}^{\alpha\,\prime\prime} = \hat{R}^{\alpha}{}_{kn0}\,p^n\,p^k \;+ \hat{T}^{\alpha}{}_{k0}{}'\,p^k \;+ \hat{T}^{\alpha}{}_{k\mu}\;p^k\,p^n\,(\Lambda^{\mu}{}_{n} \;+ \hat{T}^{\mu}{}_{n0}\,) \tag{51}$$

$$\hat{\mathcal{Y}}'' = \hat{T}^{0}{}_{kn}\,p^k\;\,d\,p^n \tag{52}$$

$$\hat{\mathcal{Q}}^{i\,\prime\prime} = \hat{T}^{i}{}_{kn}\;p^k\;\,d\,p^n \tag{53}$$

$$\hat{\mathcal{A}}^{\alpha}{}_{\beta}{}'' = \hat{R}^{\alpha}{}_{\beta k0}{}'p^k \;+ \hat{R}^{\alpha}{}_{\beta k\mu}\,p^k\,p^n\,(\Lambda^{\mu}{}_{n} \;+ \hat{T}^{\mu}{}_{n0}\,) \tag{54}$$

$$\hat{\mathcal{C}}^{\alpha}{}_{\beta}{}'' = \hat{R}^{\alpha}{}_{\beta kn}\,p^k\;\,d\,p^n\,. \tag{55}$$

At the third order:

$$\begin{aligned}\hat{\mathcal{F}}^{\alpha\,\prime\prime\prime} =&\hat{R}^{\alpha}{}_{kn0}{}'\,p^n\,p^k \;+ \hat{R}^{\alpha}{}_{kn\mu}\;p^n\,p^k\,p^m\,(\Lambda^{\mu}{}_{m} \;+ \hat{T}^{\mu}{}_{m0}\,)+\\ &+ \hat{T}^{\alpha}{}_{k0}{}''\,p^k \;+ 2\,\hat{T}^{\alpha}{}_{k\mu}{}'\,p^k\,p^n\,(\Lambda^{\mu}{}_{n} \;+ \hat{T}^{\mu}{}_{no}\,)+\\ &+ \hat{T}^{\alpha}{}_{k\mu}\;\hat{R}^{\mu}{}_{mn0}\;p^k\,p^n\,p^m \;+ \hat{T}^{\alpha}{}_{k\mu}\;\hat{T}^{\mu}{}_{m0}{}'\,p^k\,p^m +\\ &+ \hat{T}^{\alpha}{}_{k\mu}\;\hat{T}^{\mu}{}_{m\nu}\;p^k\,p^m\,p^m\,(\Lambda^{\nu}{}_{n} \;+ \hat{T}^{\nu}{}_{n0}\,)\end{aligned} \tag{56}$$

$$\hat{\mathcal{Y}}''' = \hat{R}^0{}_{kmn}\, p^m\, p^k\; d\,p^n\; + 2\hat{T}^0{}_{kn}{}'\, p^k\; d\,p^n\; + \hat{T}^0{}_{k\gamma}\; \hat{T}^\gamma{}_{nq}\, p^k\, p^n\; d\,p^q \tag{57}$$

$$\hat{\mathcal{Q}}^{i\,'''} = \hat{R}^i{}_{knq}\, p^n\, p^k\; d\,p^q\; + 2\hat{T}^i{}_{kn}{}'\, p^k\; d\,p^n\; + \hat{T}^i{}_{k\gamma}\; \hat{T}^\gamma{}_{nq}\, p^k\, p^n\; d\,p^q \tag{58}$$

$$\begin{aligned}\hat{\mathcal{A}}^\alpha{}_\beta{}''' =& \hat{R}^\alpha{}_{\beta k0}{}''\, p^k\; + 2\,\hat{R}^\alpha{}_{\beta k\mu}{}'\, p^k\, p^n\,(\Lambda^\mu{}_n\; + \hat{T}^\mu{}_{n0}\,) + \\ & + \hat{R}^\alpha{}_{\beta k\mu}\; \hat{R}^\mu{}_{nq0}\, p^k\, p^n\, p^q\; + \hat{R}^\alpha{}_{\beta k\mu}\; \hat{T}^\mu{}_{n0}{}' p^k\, p^n + \\ & + \hat{R}^\alpha{}_{\beta k\mu}\; \hat{T}^\mu{}_{m\nu}\; p^k\, p^m\, p^n\,(\Lambda^\nu{}_n\; + \hat{T}^\nu{}_{n0}\,)\end{aligned} \tag{59}$$

$$\hat{\mathcal{C}}^\alpha{}_\beta{}''' = 2\,\hat{R}^\alpha{}_{\beta kn}{}'\, p^k\; d\,p^n\; + \hat{R}^\alpha{}_{\beta k\delta}\; \hat{T}^\delta{}_{mq}\, p^k\, p^m\; d\,p^q\,. \tag{60}$$

In these expressions the directional derivatives $\partial^m{}_\lambda$ for the appropriate order m are to be expressed in terms of derivatives with respect to x^k:

$$\partial^m{}_\lambda = p^{i_1}\, p^{i_2} \dots p^{i_m}\, \partial_{i_1}\partial_{i_2} \dots \partial_{i_m} \tag{61}$$

and the result of applying $\partial^m{}_\lambda$ evaluated on $\sigma(t)$. The metric tensor

$$g = -e^0 \otimes e^0 + e^k \otimes e^k \tag{62}$$

is now given in terms of the above as a Taylor series to order λ^3, and generalises the results of [28, 29, 30, 31].

5 Identities from Gauge Covariance

Attention is restricted to background gravitational and electromagnetic fields. The former is described in terms of a metric tensor g and a metric-compatible connection ∇. The latter is given in terms of a 2-form F. The basic gravitational variables will be a class of *arbitrary* local orthonormal 1-form coframes $\{e^\alpha\}$ on spacetime related by $SO(1,3)$ transformations. In such frames the connection ∇ is represented by the spacetime 1-forms $\{\omega^\alpha{}_\beta\}$. The electromagnetic $U(1)$ connection can be represented locally by a spacetime 1-form A and $F = d\,A$. All other (matter) fields are denoted collectively by Φ. It is assumed that in a background electromagnetic and Einstein-Cartan gravitational field described by $\{e, \omega\}$ the matter field interactions can be derived from some action functional:

$$S[e, \omega, A, \Phi] \equiv \int_M \Lambda \tag{63}$$

where the 4-form Λ is constructed from the field variables $\{e, \omega, A, \Phi\}$ and their derivatives, so that S is invariant under change of local $U(1)$ and Lorentz group transformations [1]. Its invariances impose restrictions on the variational derivatives that arise in the expression

$$\int_M \mathcal{L}_X \Lambda = \int_M (\tau_\mu \wedge \mathcal{L}_X e^\mu + S_\mu{}^\nu \wedge \mathcal{L}_X \omega^\mu{}_\nu + j \wedge \mathcal{L}_X A + \mathcal{E} \wedge \mathcal{L}_X \Phi) \quad (64)$$

where the vector field X generates a local spacetime diffeomorphism and $\mathcal{L}_X$ denotes the Lie derivative with respect to X. Since $\mathcal{L}_X \Lambda = i_X \, d\Lambda + \, d\, i_X \Lambda$ and $d\Lambda = 0$ the left hand side of (64) is zero if X has compact support. The 3-forms $\tau_\mu, S_\mu{}^\nu$, and j are the basic variables in our balance laws and may be termed the source currents for gravitation and electromagnetism. In the absence of electrically charged matter and a background electromagnetic field a recent derivation of the identities that arise from gauge covariance can be found in [32]. The inclusion of electromagnetic interactions demands only a minor modification to these arguments but gives rise to additional terms. A sketch of the arguments follows.

Since the matter field equations are to be satisfied, $\mathcal{E} = 0$. Using the identity

$$\mathcal{L}_X A = i_X \, dA + \, d\, i_X A \quad (65)$$

and recognising that under a $U(1)$ gauge transformation $A \mapsto A + \, df$ for some spacetime scalar field f one may write

$$\mathcal{L}_X A = i_X F + \delta_{U(1)} A \quad (66)$$

in terms of the generator $\delta_{U(1)}$ of a $U(1)$ gauge transformation. In a similar way with the aid of the structure equations (36), (37) the terms $\mathcal{L}_X e^\mu$ and $\mathcal{L}_X \omega^\mu{}_\nu$ can be written in terms of the generators of local Lorentz transformations and exterior covariant derivatives [23]:

$$\int_M \mathcal{L}_X \Lambda = \int_M (\tau_\mu \wedge i_X T^\mu + \tau_\mu \wedge D(i_X e^\mu) + S_\mu{}^\nu \wedge i_X R^\mu{}_\nu \quad (67)$$

$$+ j \wedge i_X F + j \wedge \delta_{U(1)} A - \tau_\mu \wedge \delta_{SO(3,1)} e^\mu - S_\mu{}^\nu \wedge \delta_{SO(1,3)} \omega^\mu{}_\nu). \quad (68)$$

By gauge invariance of the action $\int_M \Lambda$

$$\int_M (\delta_{U(1)} + \delta_{SO(1,3)}) \Lambda = 0 \quad (69)$$

[1] In theories with gravitational scalar fields it is possible to extend the local gauge symmetries to include Weyl scale covariance and an associated connection 1-form.

so the last three terms in (68) are zero. If the arbitrary components $i_X e^\mu$ of X have compact support one may replace the second term in (68) by $D\tau_\mu \wedge i_X e^\mu$ and conclude:

$$\int_M (D\tau_\alpha + \tau_\mu \wedge i_{X_\alpha} T^\mu + S_\mu{}^\nu \wedge i_{X_\alpha} R^\mu{}_\nu + j \wedge i_{X_\alpha} F)\, \mathcal{W}^\alpha = 0 \tag{70}$$

where $\{X_\alpha\}$ is the dual frame and $\{\mathcal{W}^\mu\}$ a set of test 0-forms. Similarly invariance of the action $\int_M \Lambda$ under $\delta_{SO(1,3)}$ alone gives:

$$\int_M (DS_\mu{}^\nu - \frac{1}{2}(\tau_\mu \wedge e^\nu - \tau^\nu \wedge e_\mu))\, \mathcal{W}^\mu{}_\nu = 0 \tag{71}$$

for a set of test 0-forms $\{\mathcal{W}^\mu_\nu\}$. For smooth tensors on a regular domain of M one deduces the local identities:

$$D\tau_\alpha + \tau_\mu \wedge i_{X_\alpha} T^\mu + S_\mu{}^\nu \wedge i_{X_\alpha} R^\mu{}_\nu + j \wedge i_{X_\alpha} F = 0 \tag{72}$$

$$DS_\mu{}^\nu = \frac{1}{2}(\tau_\mu \wedge e^\nu - \tau^\nu \wedge e_\mu). \tag{73}$$

In these expressions

$$D\,\tau^\alpha \equiv \, d\,\tau^\alpha + \omega^\alpha{}_\beta \wedge \tau^\beta$$

$$D\,S^{\alpha\beta} \equiv \, d\,S^{a\beta} + \omega^\alpha{}_\gamma \wedge S^{\gamma\beta} + \omega^\beta{}_\gamma \wedge S^{\alpha\gamma}.$$

Finally invariance of the action under $\delta_{U(1)}$ alone yields:

$$\int_M \delta_{U(1)}\Lambda = \int_M j \wedge \delta_{U(1)} A = \int_M j \wedge \, d\, f = \int_M f\, d\, j = 0. \tag{74}$$

For smooth j on a regular domain

$$d\, j = 0 \tag{75}$$

since f, with compact support, is arbitrary.

It should be stressed that (72), (73), (75) are *identities* in a field theory satisfying the conditions used in their derivation. However in a dynamical scheme that attempts to approximate the currents $\tau_\alpha, S_\mu{}^\nu, j$ they provide powerful constraints. Thus (75) is clearly a conservation law (in an arbitrary spacetime background), since for any 4 dimensional domain $\mathcal{M} \subset M$ with $\partial\mathcal{M} = \Sigma_1 - \Sigma_2 + \mathcal{C}$ and Σ_1, Σ_2 disjoint spacelike hypersurfaces in M, and j regular on $\mathcal{M}$:

$$0 = \int_{\mathcal{M}} d\, j = \int_{\partial\mathcal{M}} j = \int_{\Sigma_1} j - \int_{\Sigma_2} j \tag{76}$$

if $\int_{\mathcal{C}} j = 0$. The integral $\int_{\Sigma_1} j$ is the electric charge content of j in Σ_1.

6 Models for the Source Currents

The balance laws (72), (73), (75) above are devoid of predictive content without further information that relates the source currents to each other and the background fields. Such *equations of state* serve to define the dynamical variables in the theory and reduce the number of degrees of freedom subject to temporal evolution. The established laws of Newtonian and special relativistic field-particle interactions offer valuable guides since they should be recovered in appropriate limits. Indeed an effective source model should enable one to recover the framework of non-relativistic continuum mechanics from the balance laws in a Minkowski spacetime free of curvature and torsion. More generally it is known how to model a spinless pressureless fluid in a pseudo-Riemannian spacetime with curvature and recover the motion of dust as geodesics associated with the Levi-Civita connection [33, 34]. In spaces with curvature and torsion analogous equations for spinning hyperfluids can be constructed [35, 36, 37, 38, 39, 40].

In [12] Dixon considered the possibility of using the pseudo-Riemannian balance laws to construct a multipole analysis of the dynamics of an extended relativistic structure. Following work by Pryce [3] he gave cogent arguments for a type of subsidiary condition that eliminated peculiar motions of electrically neutral mass monopoles in the absence of gravitation. The approach advocated here is based on a similar description of a particle in terms of the evolution of appropriate multipoles along the timelike worldline σ. The structure of the particle is therefore given in terms of components of the source currents in the Fermi frame $\mathcal{F}$. Thus we seek a consistent dynamical scheme that will determine these components and the history σ where the balance laws are satisfied to some order in a suitable multipole expansion.

The basic assumption is that the source currents for a classical particle are localised on the tubular region $\mathcal{U}$ about σ. Thus if $\Sigma_t \subset \mathcal{U}$ is the hypersurface $t =$constant then the test functions in (72), (73), (75) are rapidly decreasing as a function of the *geodesic distance* λ in the 3-ball Σ_t centred on $\sigma(t)$[2]. The precise structure of the stress-energy 3-form currents τ^α defines the nature of "particle" variables used to pass from a field to a particle description. The corresponding choice of stress-energy tensor field $T = \star^{-1}\tau_\alpha \otimes e^\alpha$ is analogous to a choice of constitutive relation in continuum mechanics relating configuration stresses and powers to strains and work. The operator $\star$ here denotes the Hodge map[3] with respect to the metric g. The choice should naturally have the proper Newtonian limit for a spinless particle. We shall also ensure that the Matthisson-Papapetrou equations for electrically neutral spinning particles emerge in a torsion-free pseudo-

[2] To lowest non-trivial order the calculations below are insensitive to the precise fall off with λ

[3] In terms of an orthonormal basis the volume form $\star 1$ is chosen as $e^0 \wedge e^1 \wedge e^2 \wedge e^3$.

Riemannian spacetimes [41, 4]. Thus in the absence of self-fields (see later) consider:

$$\tau^\alpha = P^\alpha \star e^0 + \chi F^{\alpha\beta}\,\Sigma_{\beta\mu}\star e^\mu \tag{77}$$

$$S_\mu{}^\nu = \Sigma_\mu{}^\nu \star e^0 \tag{78}$$

$$j = \rho \star e^0. \tag{79}$$

in terms of the scalar components P^α, $\Sigma^{\alpha\beta}$,$F^{\alpha\beta}$ ρ on $\mathcal{U}$ and constant χ.

Since the background fields are assumed regular on σ the components of the curvature, torsion and F as well as P^α, $\Sigma^{\alpha\beta}$, ρ can each be expanded as a Taylor series of the form (13). The coframe e^α and connection forms $\omega^\alpha{}_\beta$ are also available as expansions in λ so the coefficient of the test function in the integrand of each balance can be expressed as a power series in λ with a multinomial dependence on p^1, p^2, p^3. To extract dynamic information one now truncates each series to some order and integrates over Σ_t. This may be achieved by writing

$$p^1 = \sin\theta\cos\phi, \quad p^2 = \sin\theta\sin\phi, \quad p^3 = \cos\theta$$

and noting that if the positive integers n_1, n_2, n_3 are all even then

$$\int_{S^2} (p^1)^{n_1}\,(p^2)^{n_2}\,(p^3)^{n_3}\,d\Omega = \frac{1}{2}\frac{\Gamma(\frac{n_1+1}{2})\Gamma(\frac{n_2+1}{2})\Gamma(\frac{n_3+1}{2})}{\Gamma(\frac{n_1+n_2+n_3}{2})}. \tag{80}$$

If any integer is odd then the integral is zero. Since

$$e^0 \wedge e^1 \wedge e^2 \wedge e^3 = \lambda^2 \sin\theta\, d\,\theta \wedge\, d\,\phi \wedge\, d\,\lambda \wedge\, d\,(ct) + O(\lambda^3)$$

the lowest order of relevance includes terms up to order λ^2. The resulting equations yield ordinary differential equations for the dynamical variables $\hat\Sigma^{\alpha\beta}, \hat P^\alpha$ since the conservation equation $d\,j = 0$ requires $q = \int_{\Sigma_t} j$ to be a constant. It is convenient to introduce new variables:

$$P^i = \hat P^i$$

$$P^0 = \hat P^0$$

$$s_i = \frac{1}{2}\epsilon_{ijk}\,\hat\Sigma^{jk}$$

$$h_k = \hat{\Sigma}^{0k}$$

$$B_i = \frac{1}{2}\epsilon_{ijk}\,\hat{F}^{jk}$$

$$E_i = \hat{F}^0{}_i$$

and denote ordinary derivatives with respect to $c\,t$ by an over-dot.

In terms of the spin 2-form

$$\mathcal{S} \equiv \frac{\Sigma_{\alpha\beta}}{2}\, e^\alpha \wedge e^\beta$$

and the form $\tilde{V} = e^0$ one has $i_V \star \mathcal{S} = \star(\tilde{V} \wedge \mathcal{S})$. Since V restricts to the tangent to σ one readily recognises that $s = s_k\, e^k = -i_V\ \star \mathcal{S}$ is dual to the Pauli-Lubanski spin vector on σ where $\hat{\mathcal{S}} = h \wedge e^0 + i_V \star s$ and the 1-forms $h = h_k\, e^k$ and s are spatial: $i_{X_0}\, h = i_{X_0}\, s = 0$. Similarly with $\hat{P} \equiv P^0\, X_0 + P^k\, X_k$ one has on σ:

$$i_{\hat{P}}\hat{\mathcal{S}} = (h_k P_k)\, e^0 - P^0\, h + \epsilon_{ijk}\, s^i\, p^j\, e^k \tag{81}$$

Then with respect to a non-rotating Fermi frame ($\Omega_k = 0$), the equations (72), (73), (75) give:

$$\dot{\mathbf{s}} = \mathbf{h} \times \mathbf{A} + \frac{\chi}{2}(\hat{\mathbf{B}} \times \mathbf{s} + \hat{\mathbf{E}} \times \mathbf{h}) \tag{82}$$

$$\dot{\mathbf{h}} = \mathbf{A} \times \mathbf{s} - \frac{1}{2}\mathbf{P} + \frac{\chi}{2}(\hat{\mathbf{B}} \times \mathbf{h} + \mathbf{s} \times \hat{\mathbf{E}}) \tag{83}$$

$$\begin{aligned} \dot{P}^0 = &- \mathbf{A} \cdot \mathbf{P} + \chi[(\mathbf{h} \cdot \hat{\mathbf{E}})^{\dot{}} + \mathbf{A} \cdot (\hat{\mathbf{B}} \times \mathbf{h}) + \hat{T}^0{}_{0k}\,(\mathbf{s} \times \hat{\mathbf{E}})_k + \\ &+ \hat{B}_i\, s_j\, \hat{T}^j{}_{0i}\ - \hat{T}^j{}_{0j}\,(\hat{\mathbf{B}} \cdot \mathbf{s}) - \hat{E}_j\, \hat{h}_k\, \hat{T}^j{}_{0k}\,] \end{aligned} \tag{84}$$

$$\begin{aligned} \left(\dot{\mathbf{P}}\right)_k = &\hat{R}_{\alpha\beta k0}\, \Sigma^{\alpha\beta} - \hat{T}^\beta{}_{0k}\ P_\beta - (\mathbf{A})_k\, P^0 - q(\hat{\mathbf{E}})_k + \\ &+ \chi[(\hat{\mathbf{B}} \times \mathbf{h})^{\dot{}}_k + (\mathbf{A})_k(\hat{\mathbf{E}} \cdot \mathbf{h}) + \\ &+ \hat{E}_i\, h^j\, \hat{T}^i{}_{kj}\ - (\hat{\mathbf{E}} \times \mathbf{s})_j\, \hat{T}^0{}_{jk}\ + (\hat{\mathbf{B}} \times \mathbf{h})_j\, \hat{T}^j{}_{0k}\ - (\hat{\mathbf{E}} \cdot \mathbf{h})\, \hat{T}^0{}_{0k} \\ &- \hat{T}^j{}_{kn}\ \hat{B}_n\, s_j + \hat{T}^j{}_{kj}\ (\hat{\mathbf{B}} \cdot \mathbf{s})]. \end{aligned} \tag{85}$$

7 Involution and Constitutive Relations

For given external fields $\hat{\mathbf{E}}, \hat{\mathbf{B}}, \hat{T}^{\alpha}{}_{\beta\mu}, \hat{R}^{\alpha}{}_{\beta\mu\nu}$ equations (82), (83), (84), (85) are 10 ordinary differential equations involving 13 functions $(\mathbf{s}, \mathbf{h}, \mathbf{A}, P^{\alpha})$ of t and the constants χ, q. Additional information is required to render these equations useful. Since radiation reactions due to *self fields* have been neglected such information should provide a means of determining the acceleration $\mathbf{A}$ of the curve σ in terms of other variables, that may themselves satisfy a differential system. Since $\mathbf{A}_k = (\nabla_{\dot{\sigma}}\dot{\sigma})_k$, the equation for σ can then be determined in any coordinate system. For consistency the final set of equations should reduce to an involutive differential system. So, suppose $\xi_A(t)$ denotes collectively the variables $(\mathbf{s}, \mathbf{h}, P^{\alpha})$ *and* the background fields whose t dependence on σ is supposed known. Write (83), (82), (84), (85) and the time derivatives of the background field components as

$$\dot{\xi}_A = W_A(\xi, \mathbf{A}, \chi, q). \tag{86}$$

Let

$$\mathcal{K}_B(\xi, q, \chi) = 0 \tag{87}$$

be a set of scalar subsidiary conditions (constitutive relations indexed by B) that is appended to the above system. Then the requirement $\dot{\mathcal{K}}_B = 0$ implies

$$\frac{\partial \mathcal{K}_b(\xi, q, \chi)}{\partial \xi_A} W_A(\xi, \mathbf{A}, \chi, q)|_{\mathcal{K}} = 0 \tag{88}$$

where the relation (87) and implicit t dependence of the background has been used. If, for any χ, q, (88) determines $\mathbf{A} = \mathbf{A}(\xi, q, \chi)$ one may insert this in (86) and solve for ξ_A given initial conditions. Such a system then determines the acceleration $\mathbf{A}$. However (88) may only determine some of the components of $\mathbf{A}$, or the solution may depend on the nature of the external fields or the values of the parameters χ, q. If no suitable choice can be found for generic external fields then $\mathcal{K}_B = 0$ is an unsuitable condition for the source model under consideration and a better condition (or model) is required. Another possibility is that (88) leads to new conditions, say $L_{B'}(\xi, \chi, q) = 0$ independent of $\mathbf{A}$. In this case $\dot{L}_{B'} = 0$ requires:

$$\frac{\partial L_{B'}}{\partial \xi_A} W_A(\xi, \mathbf{A}, \chi, q)|_{L,\mathcal{K}} = 0 \tag{89}$$

These equations are now analysed in the same way as (88) and the procedure repeated until $\mathbf{A}$ is determined for a consistent χ, q, or the process exhibits a manifest inconsistency.

For models with more parameters and more components in the source currents the strategy is similar. Finding a constitutive relation that renders a differential-algebraic-system involutive is, in general, non-trivial and

further recourse to weak field approximations may be necessary to make progress. The above strategy is based on the premiss that *radiative* or self-field contributions to the balance laws are not relevant. Such contributions are expected to yield terms containing higher derivatives of $\mathbf{A}$ (that may be representations of a differential-delay system) and the above involution analysis must be generalised accordingly. The inclusion of such terms is a challenging and important issue that will not be pursued further here.

To illustrate this procedure equations (82), (83), (84), (85) will be supplemented with the relations:

$$\left(\tau_\alpha - \chi F_\alpha{}^\beta\, \Sigma_{\beta\mu} \star e^\mu\right) \wedge \star S^{\alpha\delta} = 0. \tag{90}$$

With the source model above these become the Tulczyjew-Dixon conditions:

$$P_\alpha\, \Sigma^{\alpha\beta} = 0 \tag{91}$$

or

$$i_{\hat{P}}\, \hat{\mathcal{S}} = 0 \tag{92}$$

in terms of the spin 2-form. In terms of the particle variables on σ, (92) gives:

$$\mathbf{h}\, P^0 = \mathbf{s} \times \mathbf{P} \tag{93}$$

and this implies

$$\mathbf{P} \cdot \mathbf{h} = 0 \tag{94}$$

$$\mathbf{s} \cdot \mathbf{h} = 0. \tag{95}$$

By contrast the condition

$$i_V\, \hat{\mathcal{S}} = 0 \tag{96}$$

gives $\mathbf{h} = 0$.

An immediate consequence of (92) from (82), (83) is:

$$\left(\hat{\Sigma}_{\alpha\beta} \Sigma^{\hat{\alpha\beta}}\right)^{\cdot} = (\mathbf{s} \cdot \mathbf{s} - \mathbf{h} \cdot \mathbf{h})^{\cdot} = 0 \tag{97}$$

in any background.

8 Special Cases

It is of interest to analyse (82), (83), (84), (85) in Minkowski spacetime devoid of curvature and torsion. First consider the case where $\chi = 0$. Then

$$\dot{\mathbf{s}} = \mathbf{h} \times \mathbf{A} \tag{98}$$

$$\dot{\mathbf{h}} = \mathbf{A} \times \mathbf{s} - \frac{1}{2}\mathbf{P} \tag{99}$$

$$\dot{P}^0 = -\mathbf{P} \cdot \mathbf{A} \tag{100}$$

$$\dot{\mathbf{P}} = -P^0\mathbf{A} - q\,\hat{\mathbf{E}}. \tag{101}$$

These immediately imply:

$$(\mathbf{s} \cdot \mathbf{s} - \mathbf{h} \cdot \mathbf{h})^{\dot{}} = \mathbf{h} \cdot \mathbf{P} \tag{102}$$

$$(-(P^0)^2 + \mathbf{P} \cdot \mathbf{P})^{\dot{}} = -2q\,\mathbf{P} \cdot \hat{\mathbf{E}}. \tag{103}$$

If the spin current is zero, $\mathbf{s} = 0$ and $\mathbf{h} = 0$ (so $\Sigma_{\alpha\beta}\,\Sigma^{\alpha\beta} = 0$), then the above equations reduce to

$$\mathbf{P} = 0 \tag{104}$$

$$\dot{P}^0 = 0 \tag{105}$$

and

$$\mathbf{A} = -q\,\frac{\hat{\mathbf{E}}}{P^0} \tag{106}$$

without further conditions. Introducing the constant of motion $P^0 = m$ the covariant equation of motion for the particle follows from the last equation:

$$\nabla_{\dot{\sigma}}\dot{\sigma} = \frac{q}{m}\widetilde{i_{\dot{\sigma}}F} \tag{107}$$

in terms of the Levi-Civita connection ∇ with $\tilde{\alpha} = g^{-1}(\alpha, -)$ for any 1 form α. One recognises the equation of motion of a spinless particle with mass m and electric charge q [48]. If $F = 0$ the particle follows a geodesic of this ∇.

Suppose next that $\Sigma_{\alpha\beta}\,\Sigma^{\alpha\beta} \neq 0$. The early descriptions of spinning point particles (see also [1, 41]) employed conditions leading to $\mathbf{h} = 0$ or $i_V\,\mathcal{S} = 0$. With the equations (98), (99), (100), (101) this yields:

$$\dot{\mathbf{s}} = 0 \tag{108}$$

$$\mathbf{P} = 2\mathbf{A} \times \mathbf{s} \tag{109}$$

$$\dot{P}^0 = 0 \tag{110}$$

$$\dot{\mathbf{P}} = -P^0\,\mathbf{A} - q\,\hat{\mathbf{E}}. \tag{111}$$

Then

$$(\mathbf{s}\cdot\mathbf{s})^{\dot{}} = 0 \tag{112}$$

$$(P_\alpha\,P^\alpha)^{\dot{}} = -2q\,\mathbf{P}\cdot\hat{\mathbf{E}} \tag{113}$$

and $\mathbf{s} = \mathbf{s}_0 \neq 0$ and $P^0 = m \neq 0$ are constants of the motion. Differentiating (109) gives

$$2\dot{\mathbf{A}} \times \mathbf{s}_0 = -m\mathbf{A} - q\,\hat{\mathbf{E}}. \tag{114}$$

If $q = 0$ this implies $\mathbf{A}\cdot\mathbf{s}_0 = 0$ and hence $\dot{\mathbf{A}}\cdot\mathbf{s}_0 = 0$. Taking the cross product of (114) (with $q = 0$) with $\mathbf{s}_0$ then gives:

$$\dot{\mathbf{A}} = \boldsymbol{\Omega}_0 \times \mathbf{A} \tag{115}$$

where $\boldsymbol{\Omega}_0 = -\frac{m}{2|\mathbf{s}_0|^2}\mathbf{s}_0$. Thus, in addition to geodesic motion given by $\mathbf{A} = 0$, if $\mathbf{A}(t_0) \neq 0$ for any t_0, solutions exist where the acceleration exhibits a rotation with angular velocity $\boldsymbol{\Omega}_0$:

$$\mathbf{A}(t) = \mathcal{R}(t)\,\mathbf{A}(0) \tag{116}$$

where $\mathcal{R}$ is a time dependent $SO(3)$ matrix describing a rotation with angular velocity $\boldsymbol{\Omega}_0$. Such a motion is unnatural for a classical particle free of electromagnetic and gravitational interactions. The existence of such solutions is one of the reasons that the Tulczyjew-Dixon condition is to be

preferred over (96). If $q \neq 0$ then the additional terms yield the modified equation

$$\dot{\mathbf{A}} = \mathbf{\Omega}_0 \times \mathbf{A} - \frac{q}{m}\hat{\mathbf{E}} \times \mathbf{\Omega}_0 - 4\frac{q}{m}\mathbf{\Omega}_0\,(\dot{\hat{\mathbf{E}}} \cdot \mathbf{\Omega_0})\left(\frac{|\mathbf{s}_0|^2}{m^2}\right) \tag{117}$$

which also exhibits unnatural motions requiring a specification of an initial acceleration.

With (92) instead of (96) equations (98), (99), (100), (101) reduce to a system that determines $\mathbf{A}$ rather than $\dot{\mathbf{A}}$ and hence fixes the motion of the particle in terms of its initial position and velocity. Differentiating the condition (93) and using equations (98), (99), (100), (101) yields a new (vector) condition:

$$P^0\mathbf{P} = 2q\mathbf{s} \times \hat{\mathbf{E}} \tag{118}$$

and hence $\mathbf{s} \cdot \mathbf{P} = 0$ and $\hat{\mathbf{E}} \cdot \mathbf{P} = 0$. The invariant $P_\alpha\, P^\alpha$ is a constant of the motion which we take to be the non-zero constant $-m^2$. Differentiating (118) and using the previously obtained equations gives rise to the equation:

$$(P^0)^4\mathbf{A} + (P^0)^3\, q\,\hat{\mathbf{E}} + 2q\,(P^0)^2\,\mathbf{s} \times \dot{\hat{\mathbf{E}}} - 4q^2\,(\mathbf{A} \cdot \mathbf{s})\,\hat{\mathbf{E}} \times \mathbf{r} = 0 \tag{119}$$

where $\mathbf{r} \equiv \mathbf{s} \times \hat{\mathbf{E}}$. For a generic $\hat{\mathbf{E}}$, ($\mathbf{r} \neq 0$), one may take $\mathbf{r}, \mathbf{s}, \hat{\mathbf{E}}$ as a basis in which to evaluate $\mathbf{A}$ from (119) in terms of $P^0, \mathbf{s}, \hat{\mathbf{E}}, \dot{\hat{\mathbf{E}}}$. Thus

$$\mathbf{A} = A_r\,\mathbf{r} + A_s\,\mathbf{s} + A_E\,\hat{\mathbf{E}} \tag{120}$$

where

$$A_r = \frac{2q}{|\mathbf{r}|^2\,P^{0 2}}\,\mathbf{r} \cdot (\dot{\hat{\mathbf{E}}} \times \mathbf{s}) \tag{121}$$

$$A_E = \frac{|\mathbf{s}|^2}{|\mathbf{r}|^2}\,\mathbf{A} \cdot \hat{\mathbf{E}} - \frac{(\hat{\mathbf{E}} \cdot \mathbf{s})}{|\mathbf{r}|^2}\mathbf{A} \cdot \mathbf{s} \tag{122}$$

$$\mathbf{A}_s = -\frac{(\hat{\mathbf{E}} \cdot \mathbf{s})}{|\mathbf{r}|^2}\,\mathbf{A} \cdot \hat{\mathbf{E}} + \frac{|\hat{\mathbf{E}}|^2}{|\mathbf{r}|^2}\,\mathbf{A} \cdot \mathbf{s} \tag{123}$$

and

$$\mathbf{A} \cdot \hat{\mathbf{E}} = -\frac{q|\hat{\mathbf{E}}|^2}{(P^0)} - \frac{2q}{(P^0)^2}\,\hat{\mathbf{E}} \cdot (\mathbf{s} \times \dot{\hat{\mathbf{E}}}) \tag{124}$$

$$\mathbf{A} \cdot \mathbf{s} = -\frac{qP^0(\hat{\mathbf{E}} \cdot \mathbf{s})}{\Big((P^0)^2 + 2q\,(\hat{\mathbf{E}} \cdot \mathbf{h})\Big)} \tag{125}$$

with

$$\mathbf{h} = \frac{2q}{(P^0)^2}\,\mathbf{s}\times(\mathbf{s}\times\hat{\mathbf{E}}). \tag{126}$$

The above expression for $\mathbf{A}(P^0, \mathbf{s}, \hat{\mathbf{E}}, \dot{\hat{\mathbf{E}}})$ can now be inserted into (98), (100) yielding:

$$\dot{P}^0 = -\frac{2q}{P^0}\,\hat{\mathbf{E}}\cdot(\mathbf{A}\times\mathbf{s}) \tag{127}$$

$$\dot{\mathbf{s}} = -\frac{2q}{(P^0)^2}\mathbf{A}\times(\mathbf{s}\times(\mathbf{s}\times\hat{\mathbf{E}})). \tag{128}$$

Given initial values for $\mathbf{s}$ and P^0 solutions of the coupled equations (127) and (128) can be used to determine the acceleration from (120). Since $\mathbf{h}=0$ and $\dot{\mathbf{s}}=0$ if $\hat{\mathbf{E}}=0$ we set $\mathbf{s}|_{\hat{\mathbf{E}}=0}=\mathbf{s}_0$ constant. Thus the constant of motion $\mathbf{s}\cdot\mathbf{s}-\mathbf{h}\cdot\mathbf{h}=|\mathbf{s}_0|^2$ identifies $\mathbf{s}_0$ as the classical spin of the particle in the absence of electromagnetic interactions. Similarly m may be identified with the particle's rest mass in the absence of such interactions. Thus at any $t=t_0$:

$$\left(P^0(t_0)\right)^2 = m^2 + \frac{4q^2}{P^0(t_0)^2}\,|\mathbf{s}\times\hat{\mathbf{E}}|^2(t_0) \tag{129}$$

$$|\mathbf{s}(t_0)|^2 = |\mathbf{s}_0|^2 + \frac{4q^2}{P^0(t_0)^4}\,|(\mathbf{s}\times(\mathbf{s}\times\hat{\mathbf{E}}))|^2(t_0). \tag{130}$$

If the particle moves slowly with velocity $\mathbf{v}$ in an inertial Minkowski frame, where the electromagnetic field has components $\mathcal{E}=\mathbf{0},\mathcal{B}$, [43], then in the Fermi frame of the particle $\hat{\mathbf{E}}\simeq\frac{\mathbf{v}}{c}\times\mathcal{B}$, $\hat{\mathbf{B}}\simeq\mathcal{B}$. Thus the spin coupling to the electromagnetic field indicates that it does not have a magnetic moment. Such an interaction arises from the inclusion of the terms in the current that depend on χ. To see this consider $\chi << 1$. Then the equations (98), (99), (100), (101) are replaced by:

$$\dot{\mathbf{h}} = -\frac{\mathbf{P}}{2} + \mathbf{A}\times\mathbf{s} - \frac{\chi}{2}\left(\hat{\mathbf{E}}\times\mathbf{s} + \mathbf{h}\times\hat{\mathbf{B}}\right) \tag{131}$$

$$\dot{\mathbf{s}} = \mathbf{h}\times\mathbf{A} + \frac{\chi}{2}\left(\hat{\mathbf{B}}\times\mathbf{s} + \hat{\mathbf{E}}\times\mathbf{h}\right) \tag{132}$$

$$\dot{P}^0 = -\mathbf{P}\cdot\mathbf{A} + \chi\left(\hat{\mathbf{E}}\cdot(\mathbf{A}\times\mathbf{s}) - \frac{1}{2}\,\mathbf{P}\cdot\hat{\mathbf{E}}\right) + O(\chi^2) \tag{133}$$

$$\dot{\mathbf{P}} = -q\,\hat{\mathbf{E}} - P^0\mathbf{A} + \chi\left(\hat{\mathbf{B}}\times(\mathbf{A}\times\mathbf{s}) + \mathbf{A}\,(\hat{\mathbf{E}}\cdot\mathbf{h}) - \frac{1}{2}\,\hat{\mathbf{B}}\times\mathbf{P}\right) + O(\chi^2). \tag{134}$$

Consider static fields ($\dot{\hat{\mathbf{E}}} = \mathbf{0}$, $\dot{\hat{\mathbf{B}}} = \mathbf{0}$). Then after differentiating the condition (92), taking scalar products of the result with $\mathbf{s}$ and $\mathbf{P}$ and using the above equations along with the relations $\mathbf{P}\cdot\mathbf{s} = 0$ and $\mathbf{s}\cdot\mathbf{h} = 0$ an additional constraint arises that for $P^0 \neq 0$ requires $\mathbf{P} = 0$ at this order. This in turn implies (for $\mathbf{s} \neq 0$) $\mathbf{h} = 0$ and hence $P^0 = m$ constant. Furthermore now:

$$\mathbf{A} = -\frac{q}{m}\,\hat{\mathbf{E}} \tag{135}$$

$$\dot{\mathbf{s}} = \frac{\chi}{2}\,\hat{\mathbf{B}}\times\mathbf{s}. \tag{136}$$

But since $\dot{\mathbf{h}} = 0$, $\mathbf{A}\times\mathbf{s} = \chi\,(\hat{\mathbf{E}}\times\mathbf{s})/2$ or

$$\chi = -\frac{2q}{m}. \tag{137}$$

Thus, in this small χ, stationary field, limit one has (107) and

$$\dot{\hat{\mathcal{S}}}_{\alpha\beta} = \frac{1}{2}\,\hat{F}_{\alpha\mu}\hat{\mathcal{S}}^{\mu}{}_{\beta} \tag{138}$$

and one may identify the interaction of electromagnetism with a charged particle possessing a magnetic moment related to its spin with a gyromagnetic ratio 2 [49]. Thus the source currents (142), (143), (144), are appropriate for the description of a spinning particle of mass m, electric charge q and gyromagnetic coupling defined by (137).

9 Generalisations

A relativistic description of a material rod involves a 2-dimensional immersed timelike submanifold Σ of spacetime (the rod *worldsheet*) with attached material body frames that relate dilation, bending, shear and torsional deformations to the immersion. As before, to obtain such an immersion from an interacting field description requires a reduction, or truncation, of field degrees of freedom. This is most naturally accomplished in terms of a field

of local adapted ortho-normal spacetime frames generated by the transport of local material body frames along spacelike autoparallels of the Lorentz connection ∇ to points in a neighbourhood $\mathcal{U}$ of the worldsheet Σ. Dimensional reduction of the stress-energy and spin Bianchi identities, and entropy imbalance relations on $\mathcal{U}$, is performed in new Fermi coordinates $\{\sigma_0, \sigma_1, \lambda, p_1, p_2\}$ (with $p_1^2 + p_2^2 = 1$) adapted to such body frames. Their definition in terms of the exponential maps is a generalisation of that given earlier with Σ replacing the worldline σ. In such coordinates the history of the rod Σ is given as the timelike submanifold $\{\lambda p_1 = 0, \lambda p_2 = 0\}$ for some range of $\{\sigma_0, \sigma_1\}$ and the local coframe on $\mathcal{U}$ may be written:

$$e^j = f^j{}_i \, d\sigma^i + \phi^j \tag{139}$$

$$e^\beta = f^\beta{}_i \, d\sigma^i + p^\beta \, d\lambda + \phi^\beta \tag{140}$$

for $i = 0, 1$ and $\alpha, \beta = 2, 3$, where $f^j{}_i$, $f^\beta{}_i$ are 0 forms dependent on σ^i, λ, p^α and with 1-forms ϕ^A $(A = 0, 1, 2, 3)$ independent of $d\sigma^i$, $d\,\lambda$ but components dependent on σ^i, λ, p^α. Similarly in such Fermi coordinates, one may write the connection 1-forms in this coframe

$$\omega^A{}_B = \mathcal{A}^A{}_{Bi} \, d\sigma^i + \mathcal{C}^A{}_B \tag{141}$$

where $\mathcal{A}^A{}_{Bi}$ are 0 forms dependent on $\sigma,^i$ λ, p^α and $\mathcal{C}^A{}_{Bi}$ are 1-forms independent of $d\sigma^i$, $d\lambda$ but with components dependent on σ^i, λ, p^α. All the functions appearing in these expressions can be found as before by solving the structure equations that define the torsion and curvature of ∇.

The source currents τ_A, S^{AB} and j are now localised on the tubular region $\mathcal{U}$ about the rod history Σ. The parameterisation of the stress-energy 3-form currents τ^A, S^{AB} on Σ defines the rod variables used to pass from a field to a " Cosserat rod " description [4]. In the absence of dissipation one can generate relativistic Cosserat models interacting with gravitational and electromagnetic fields from:

$$\tau^A = P^A{}_i \star e^i + \chi F^{AB} \, \Sigma_{BB'} \star e^{B'} \tag{142}$$

$$S_A{}^B = \Sigma_A{}^B{}_j \star e^j \tag{143}$$

$$j = \rho \star e^0 + J \star e^1. \tag{144}$$

in terms of the scalar components $P^A{}_i$, $\Sigma^{AB}{}_j$, F^{AB}, ρ, J on $\mathcal{U}$ and constant χ. For thermally active media one must supplement these equations with phenomenological consitutive relations for the stress-energy, spin, entropy and heat forms subject to the constraints imposed by entropy production.

[4] Strings arise as a special case of shear-free rods

10 Conclusions

With the aid of the structure equations for a metric-compatible connection on the bundle of analytic orthonormal frames over a tubular domain of spacetime, a method for constructing a local section by Taylor series has been described. Explicit computations have been presented in terms of Fermi coordinates for this domain associated with a (generalised) Fermi-Walker frame attached to a timelike curve. This tubular section is used to effect a truncation scheme based on balance laws associated with identities derived from the invariances of an action integral. The curve is defined as the history of a massive particle with spin. A series of auxiliary *constitutive* relations is used to construct models for the stress-energy and electromagnetic currents that enter into these balance laws. Taylor series representations are expressed in terms of expansions in the arc distance along spacelike autoparallels associated with the Fermi-Walker frame. By suitably truncating these series one may construct models of spinning matter. In the absence of self-forces the determination of a consistent dynamical scheme for the acceleration of the particle worldline can be effected by ensuring that the auxiliary conditions render each truncated set an involutive system of ordinary differential equations. This procedure has been demonstrated by analysing the dynamical equations in Minkowski spacetime where it becomes possible to identify the expected interactions of a charged spinning particle with a particular magnetic moment in an external electromagnetic field.

A number of interesting features emerges from this investigation. The inclusion of charged matter interactions with gravity yields terms that may be interpreted as tidal forces that couple components of the torsion tensor to the electromagnetic field via the particle spin. Such interactions are novel and may offer new experimental probes for the detection of spacetime torsion. Another feature is that in the lowest approximation considered here, electrically neutral massive *spinless* particles are predicted to follow *autoparallels* of the Levi-Civita connection. By contrast the precession rate and motion of spinning gyroscopes in gravitational fields with torsion differ from those in a torsion-free environment [45]. The differences can be estimated from the results given in this paper.

The approach adopted here offers a number of natural generalisations. Fermi charts adapted to spacetime submanifolds with co-dimension less than three permit the methods to be applied to derive the relativistic motion of Cosserat media [46] in general spacetimes. The associated orthonormal coframe and source currents for a relativistic rod have been mentioned in IX.

The above analysis has been conservative. Prompted by puzzles in cosmology many modern theories of gravitation invoke scalar fields that couple directly to spacetime geometry and Yang-Mills fields. Some of these approaches suggest that at some scale the dilation group may arise as an ad-

ditional local symmetry. If this group is included it is possible to generalise the balance laws to incorporate fields with Weyl charge in non-Riemannian geometries. It is then possible that not all electrically neutral *spinless* particles have the same type of worldline. In geometries with torsion, autoparallel worldlines are in general distinct from geodesics of the (torsion-free) Levi-Civita connection [44]. Thus the shift in perihelia of celestial objects in highly eccentric orbits may also be used as probes in generalised theories of gravitation [15].

A discussion of the significance of radiative forces on the motion of charged particles in geometries with torsion and an extension of the methods to retarded null coordinates associated with an arbitrary timelike curve will be presented elsewhere.

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